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SEXTIC FORMALISM IN ANISOTROPIC ELASTICITY FOR ALMOST NON-SEMISIMPLE MATRIX N

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Abstract—The sextic formalism of Stroh for anisotropic elasticity leads to the eigen-relation $\mathbf{N}\boldsymbol{\xi} = p\boldsymbol{\xi}$ in which N is a 6×6 real matrix. The orthogonality and closure relations as well as many other relations involving the eigenvalues p and the eigenvectors $\boldsymbol{\xi}$ are based on the assumption that N is simple or semisimple so that the six eigenvectors $\boldsymbol{\xi}_\alpha$ span a six-dimensional space. Problems arise when N is non-semisimple. In fact there are problems even when N is almost non-semisimple. We present a modified formalism which is valid regardless of whether N is simple, almost non-semisimple or non-semisimple. The modified formalism does not apply when N is semisimple.

1. INTRODUCTION

The sextic formalism for anisotropic elasticity originally due to Stroh[1, 2] assumes that the 6×6 real matrix N is simple. This means that the eigenvalues p_α $(\alpha = 1, 2, \ldots, 6)$ of N are distinct so that there are six independent eigenvectors $\boldsymbol{\xi}_\alpha$. The formalism applies also to semisimple N in which there is a repeated eigenvalue, say $p_1 = p_2$, but there exist two independent eigenvectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$. When N is non-semisimple, i.e. when $p_1 = p_2$ and there exists only one independent eigenvector associated with p_1 and p_2, the Stroh formalism does not apply. Anisotropic elastic materials which lead to a non-semisimple N are called degenerate materials. Isotropic materials are a special group of degenerate materials. Nishioka and Lothe[3, 4] studied the limiting behavior of the Stroh formalism when the material becomes isotropic. Lothe and Barnett[5] and Chadwick and Smith[6] introduce the generalized eigenvectors and obtain an important result that some relations for simple N continue to hold for non-semisimple N if the eigenvectors are replaced by the generalized eigenvectors. However, as we will see in this paper, not all relations for simple N can be converted to relations for non-semisimple N by simply replacing the eigenvectors by the generalized eigenvectors. Examples will be given in this paper. The main purpose of this paper however is to look at the situation in which N is almost non-semisimple.

When N is simple or semisimple, the Stroh formalism applies. When N is non-semisimple, the generalized eigenvectors take the place of eigenvectors. The transition of the formalism from a simple or semisimple to non-semisimple N is not continuous. This suggests that some difficulties may arise when N is almost non-semisimple. Indeed, as we will see in Section 2 where we summarize the Stroh formalism, when N is almost non-semisimple the magnitude of the orthonormalized eigenvectors associated with the almost equal eigenvalues is very large and becomes infinite as the two eigenvalues become equal. To overcome this difficulty we present in Section 3 a modified sextic formalism which applies to almost non-semisimple N. The formalism remains valid when N is non-semisimple. In fact the assumption of almost non-semisimple is not required in the derivation and hence the formalism applies to simple N as well. The modified formalism however does not apply to N which is semisimple.

In Section 4 we show the conversion from the Stroh formalism to the present modified formalism. With the conversion many relations which are valid for simple or semisimple N can be rewritten for non-semisimple or almost non-semisimple N. Applications to sum rules are given in Section 5. Finally we show in Section 6 how one can split the generalized 6-vectors $\boldsymbol{\xi}$ for almost non-semisimple N into two 3-vectors **a** and **b** and determine them separately.

2. THE STROH SEXTIC FORMALISM

In a fixed rectangular coordinate system (x_1, x_2, x_3) let the stress σ_{ij} and strain ε_{ij} of the material be related by

$$\sigma_{ij} = C_{ijks}\varepsilon_{ks} \tag{1}$$

$$C_{ijks} = C_{jiks} = C_{ijsk} = C_{ksij} \tag{2}$$

where C_{ijks} are the elasticity constants. Unless stated otherwise repeated indices imply summation. For two-dimensional deformations in which the displacements u_k $(k = 1, 2, 3)$, depend on x_1 and x_2 only, a general solution for u_k can be written in matrix notation as

$$\mathbf{u} = \mathbf{a}f(z) \tag{3}$$

$$z = x_1 + px_2 \tag{4}$$

in which f is an arbitrary function of z. The eigenvalue p and the eigenvector $\mathbf{a}$ are determined from[7]

$$\mathbf{D}(p)\mathbf{a} = \mathbf{0} \tag{5}$$

$$\mathbf{D}(p) = \mathbf{Q} + p(\mathbf{R}+\mathbf{R}^T) + p^2\mathbf{T} \tag{6}$$

where superscript T stands for the transpose and the 3×3 matrices $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$ are given by

$$Q_{ik} = C_{i1k1}, \qquad R_{ik} = C_{i1k2}, \qquad T_{ik} = C_{i2k2}. \tag{7}$$

Matrices $\mathbf{Q}$ and $\mathbf{T}$ are symmetric and positive definite if the strain energy is positive. Introducing the new vector

$$\mathbf{b} = (\mathbf{R}^T + p\mathbf{T})\mathbf{a} = -\frac{1}{p}(\mathbf{Q}+p\mathbf{R})\mathbf{a} \tag{8}$$

in which the second equality comes from eqn (5), the stresses are obtained from the stress function $\boldsymbol{\phi}$ by[1, 2]

$$\sigma_{i1} = -\partial\phi_i/\partial x_2, \qquad \sigma_{i2} = \partial\phi_i/\partial x_1 \tag{9}$$

$$\boldsymbol{\phi} = \mathbf{b}f(z). \tag{10}$$

Equations $(8)_1$ and $(8)_2$ can be written in the standard eigen-relation as

$$\mathbf{N}\boldsymbol{\xi} = p\boldsymbol{\xi} \tag{11}$$

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix}, \qquad \boldsymbol{\xi} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \tag{12}$$

$$\left.\begin{aligned} \mathbf{N}_1 &= -\mathbf{T}^{-1}\mathbf{R}^T, \qquad \mathbf{N}_2 = \mathbf{T}^{-1} = \mathbf{N}_2^T, \\ \mathbf{N}_3 &= \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q} = \mathbf{N}_3^T. \end{aligned}\right\} \tag{13}$$

Thus $\boldsymbol{\xi}$ is the right eigenvector of the 6×6 real matrix $\mathbf{N}$. The left eigenvector $\boldsymbol{\eta}$ satisfies

$$\mathbf{N}^T\boldsymbol{\eta} = p\boldsymbol{\eta}. \tag{14}$$

Introducing the 6×6 matrix **J** by

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \tag{15}$$

where **I** is the identity matrix, it can be shown that

$$\mathbf{JN} = (\mathbf{JN})^T = \mathbf{N}^T\mathbf{J}. \tag{16}$$

From eqns (11), (14) and (16) we may set without loss of generality

$$\boldsymbol{\eta} = \mathbf{J}\boldsymbol{\xi} = \begin{bmatrix} \mathbf{b} \\ \mathbf{a} \end{bmatrix}. \tag{17}$$

Since p cannot be real if the strain energy is positive[7], we have three pairs of complex conjugates for p as well as for $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. If p_α, $\boldsymbol{\xi}_\alpha$ and $\boldsymbol{\eta}_\alpha$ ($\alpha = 1, \ldots, 6$) are the eigenvalues and the eigenvectors, we let

$$\left.\begin{array}{ll} p_{\alpha+3} = \bar{p}_\alpha, & \text{Im}\, p_\alpha > 0 \\ \boldsymbol{\xi}_{\alpha+3} = \bar{\boldsymbol{\xi}}_\alpha, & \boldsymbol{\eta}_{\alpha+3} = \bar{\boldsymbol{\eta}}_\alpha \end{array}\right\}\alpha = 1, 2, 3 \tag{18}$$

where Im denotes the imaginary part and an overbar stands for the complex conjugate. When **N** is simple or semisimple, $\boldsymbol{\xi}_\alpha$ span a six-dimensional space and are orthogonal to $\boldsymbol{\eta}_\alpha$. Since $\boldsymbol{\xi}_\alpha$ obtained from eqn (11) are unique up to a multiplicative constant, we may normalize $\boldsymbol{\xi}_\alpha$ such that (with $\boldsymbol{\eta}_\alpha$ determined from eqn (17))

$$\boldsymbol{\eta}_\beta^T\boldsymbol{\xi}_\alpha = \delta_{\alpha\beta} \tag{19}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. The orthonormal relations can be written in matrix notation as

$$\mathbf{V}^T\mathbf{U} = \mathbf{I} \tag{20}$$

in which the 6×6 matrices **U** and **V** are

$$\left.\begin{array}{l} \mathbf{U} = [\boldsymbol{\xi}_1, \quad \boldsymbol{\xi}_2, \quad \boldsymbol{\xi}_3, \quad \bar{\boldsymbol{\xi}}_1, \quad \bar{\boldsymbol{\xi}}_2, \quad \bar{\boldsymbol{\xi}}_3] \\ \mathbf{V} = [\boldsymbol{\eta}_1, \quad \boldsymbol{\eta}_2, \quad \boldsymbol{\eta}_3, \quad \bar{\boldsymbol{\eta}}_1, \quad \bar{\boldsymbol{\eta}}_2, \quad \bar{\boldsymbol{\eta}}_3]. \end{array}\right\} \tag{21}$$

If we introduce the 3×3 matrices

$$\mathbf{A} = [\mathbf{a}_1, \quad \mathbf{a}_2, \quad \mathbf{a}_3], \qquad \mathbf{B} = [\mathbf{b}_1, \quad \mathbf{b}_2, \quad \mathbf{b}_3] \tag{22}$$

we may write **U** and **V** as, using eqns $(12)_2$ and (17)

$$\mathbf{U} = \begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix}, \qquad \mathbf{V} = \mathbf{JU}. \tag{23}$$

Equation (20) implies that $\mathbf{V}^T$ and **U** are the inverse of each other and hence the order of the product can be interchanged. We have

$$\mathbf{U}\mathbf{V}^{\mathrm{T}} = \mathbf{I} \tag{24}$$

or, carrying out the matrix multiplications using eqns (15) and (23)

$$\left.\begin{array}{l}\mathbf{A}\mathbf{A}^{\mathrm{T}}+\bar{\mathbf{A}}\bar{\mathbf{A}}^{\mathrm{T}} = \mathbf{0} = \mathbf{B}\mathbf{B}^{\mathrm{T}}+\bar{\mathbf{B}}\bar{\mathbf{B}}^{\mathrm{T}} \\ \mathbf{B}\mathbf{A}^{\mathrm{T}}+\bar{\mathbf{B}}\bar{\mathbf{A}}^{\mathrm{T}} = \mathbf{I} = \mathbf{A}\mathbf{B}^{\mathrm{T}}+\bar{\mathbf{A}}\bar{\mathbf{B}}^{\mathrm{T}}.\end{array}\right\} \tag{25}$$

These are the closure relations. Equations (25) tell us that there exist real matrices **H**, **L** and **S** such that

$$\left.\begin{array}{l}\mathbf{H} = 2i\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{H}^{\mathrm{T}} \\ \mathbf{L} = -2i\mathbf{B}\mathbf{B}^{\mathrm{T}} = \mathbf{L}^{\mathrm{T}} \\ \mathbf{S} = i(2\mathbf{A}\mathbf{B}^{\mathrm{T}}-\mathbf{I}).\end{array}\right\} \tag{26}$$

We see that **H** and **L** are symmetric, and can be shown to be positive definite if the strain energy is positive[6]. The three real matrices **H**, **L** and **S** play important roles in the problems of anisotropic elasticity and surface waves (see, e.g. Refs [6, 8–11]).

The above formalism from eqns (19) to (26) are valid if **N** is simple or semisimple because we have six independent eigenvectors $\boldsymbol{\xi}_\alpha$. If **N** is non-semisimple, say we have $p_1 = p_2$ and also $\boldsymbol{\xi}_1 = \boldsymbol{\xi}_2$, we do not have six independent eigenvectors to span the six-dimensional space. Consequently, eqns (19)–(26) are not valid. Isotropic materials are the well-known example of having a non-semisimple **N** for which $p_1 = p_2 = i$ and $\boldsymbol{\xi}_1 = \boldsymbol{\xi}_2$. In fact $p_3 = i$ also but $\boldsymbol{\xi}_3$ is independent of $\boldsymbol{\xi}_1$.

One encounters difficulties not only when **N** is non-semisimple but also when **N** is almost non-semisimple. This means that p_1 and p_2 are almost equal as are $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$. To see the problems which may arise when **N** is almost non-semisimple, let $\hat{\boldsymbol{\xi}}_1$ and $\hat{\boldsymbol{\xi}}_2$ be *unit* vectors satisfying eqn (11) for $p = p_1$ and p_2, respectively. Assuming that p_1, p_2 are almost equal as are $\hat{\boldsymbol{\xi}}_1$, $\hat{\boldsymbol{\xi}}_2$, we let

$$\hat{\boldsymbol{\xi}}_2 = \hat{\boldsymbol{\xi}}_1 + \varepsilon(\delta)\mathbf{y}, \qquad \delta = p_2 - p_1 \tag{27}$$

in which **y** is a unit vector and ε is a function of δ such that as δ approaches zero so does ε. To have an orthonormal system we set

$$\left.\begin{array}{ll}\boldsymbol{\xi}_1 = k_1\hat{\boldsymbol{\xi}}_1, & \boldsymbol{\xi}_2 = k_2\hat{\boldsymbol{\xi}}_2 = k_2(\hat{\boldsymbol{\xi}}_1+\varepsilon\mathbf{y}), \\ \boldsymbol{\eta}_1 = \mathbf{J}\boldsymbol{\xi}_1, & \boldsymbol{\eta}_2 = \mathbf{J}\boldsymbol{\xi}_2\end{array}\right\} \tag{28}$$

where k_1, k_2 are complex constants to be determined. Application of eqn (19) for $\alpha, \beta = 1, 2$, leads to

$$\left.\begin{array}{l}k_1^2\hat{\boldsymbol{\xi}}_1^{\mathrm{T}}\mathbf{J}\hat{\boldsymbol{\xi}}_1 = 1 \\ k_2^2(\hat{\boldsymbol{\xi}}_1^{\mathrm{T}}\mathbf{J}\hat{\boldsymbol{\xi}}_1 + 2\varepsilon\mathbf{y}^{\mathrm{T}}\mathbf{J}\hat{\boldsymbol{\xi}}_1 + \varepsilon^2\mathbf{y}^{\mathrm{T}}\mathbf{J}\mathbf{y}) = 1 \\ k_1k_2(\hat{\boldsymbol{\xi}}_1^{\mathrm{T}}\mathbf{J}\hat{\boldsymbol{\xi}}_1 + \varepsilon\mathbf{y}^{\mathrm{T}}\mathbf{J}\hat{\boldsymbol{\xi}}_1) = 0.\end{array}\right\} \tag{29}$$

Ignoring the ε^2 term when δ is small, we have

$$k_2^2 = -k_1^2 = (\varepsilon\mathbf{y}^{\mathrm{T}}\mathbf{J}\hat{\boldsymbol{\xi}}_1)^{-1}. \tag{30}$$

Hence k_1 and k_2 are of order $\varepsilon^{-1/2}$. Consequently, the *orthonormalized* vectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are very large vectors when δ is small and become unbounded when δ approaches zero. This creates problems for a numerical calculation of the eigenvectors when **N** is almost non-semisimple. Equations (30) also tell us that $k_2 = \pm ik_1$ and hence, as $\delta \to 0$, the orthonormalized eigenvectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are *not* exactly equal but differ by a factor of $\pm i$. The statement that ξ_1 and ξ_2 are almost equal should therefore be interpreted as almost linearly dependent.

In the next section we present a modified formalism for the case when N is almost non-semisimple. We will see in the derivation that the assumption of almost non-semisimple is unnecessary. The eigenvalues p_1 and p_2 need not be almost equal. The only requirement is that if p_1 and p_2 are almost equal, so are ξ_1 and ξ_2.

3. MODIFIED SEXTIC FORMALISM

We assume in this section that there is a possibility that p_1 and p_2 are either equal or almost equal. When that happens, we assume that ξ_1 and ξ_2 are also equal or almost equal. By eqns (18) p_4 and p_5 as well as ξ_4 and ξ_5 are equal or almost equal. It suffices to discuss the modifications required for ξ_1 and ξ_2 only.

From eqn (11) we have

$$\left.\begin{aligned} \mathbf{N}\xi_1^0 &= p_1\xi_1^0 \\ \mathbf{N}\xi_2^0 &= p_2\xi_2^0 \end{aligned}\right\} \tag{31}$$

in which ξ_1^0 and ξ_2^0 are scalar multiples of ξ_1 and ξ_2 obtained in the last section. The scalar multiples are not unity or $\pm i$ because of a different orthonormal system we are introducing here. Instead of eqns (31) we consider

$$\left.\begin{aligned} \mathbf{N}\xi_1^0 &= p_1\xi_1^0 \\ \mathbf{N}\xi_2' &= p_2\xi_2'+\xi_1^0 \end{aligned}\right\} \tag{32}$$

where

$$\left.\begin{aligned} \xi_2' &= (\xi_2^0-\xi_1^0)/\delta \\ \xi_2^0 &= \xi_1^0+\delta\xi_2' \end{aligned}\right\} \tag{33}$$

$$\delta = p_2-p_1. \tag{34}$$

Equation $(32)_2$ is obtained when we subtract eqn $(31)_1$ from eqn $(31)_2$ and divide the resulting equation by (p_2-p_1). Likewise, we will consider for the left eigenvectors the following equations:

$$\left.\begin{aligned} \mathbf{N}^T\eta_1' &= p_1\eta_1'+\eta_2^0 \\ \mathbf{N}^T\eta_2^0 &= p_2\eta_2^0 \end{aligned}\right\} \tag{35}$$

in which

$$\left.\begin{aligned} \eta_1' &= (\eta_2^0-\eta_1^0)/\delta \\ \eta_1^0 &= \eta_2^0-\delta\eta_1'. \end{aligned}\right\} \tag{36}$$

Thus instead of $\xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0$, we will use $\xi_1^0, \xi_2', \eta_1', \eta_2^0$. They are determined from eqns (32) and (35). The vectors ξ_2^0 and η_1^0 are not employed, but their relations with $\xi_1^0, \xi_2', \eta_1', \eta_2^0$ as given by eqns (33) and (36) will be useful in establishing certain identities. Hence δ can be arbitrary, zero or non-zero. Instead of solving eqn (35) for η_1' and η_2^0, they can be obtained from ξ_1^0 and ξ_2' by applying eqns (17) and (36). We have

$$\left.\begin{aligned}\boldsymbol{\eta}'_1 &= \mathbf{J}\boldsymbol{\xi}'_2\\ \boldsymbol{\eta}_2 &= \mathbf{J}\boldsymbol{\xi}_1+\delta\mathbf{J}\boldsymbol{\xi}'_2.\end{aligned}\right\} \tag{37}$$

The new vectors satisfy the following relations:

$$\boldsymbol{\eta}_2^{\mathrm{T}}\boldsymbol{\xi}'_2-\boldsymbol{\eta}'^{\mathrm{T}}_1\boldsymbol{\xi}_1 = \delta\boldsymbol{\eta}'^{\mathrm{T}}_1\boldsymbol{\xi}'_2 \tag{38}$$

$$\boldsymbol{\eta}_2^{\mathrm{T}}\boldsymbol{\xi}_1 = 0. \tag{39}$$

Equations (38) and (39) are obtained when we pre-multiply eqns $(32)_2$ and $(32)_1$, respectively, by $\boldsymbol{\eta}'^{\mathrm{T}}_1$ and use eqn $(35)_1$. To form an orthonormal system we must have

$$\boldsymbol{\eta}'^{\mathrm{T}}_1\boldsymbol{\xi}_1 = 1, \qquad \boldsymbol{\eta}_2^{\mathrm{T}}\boldsymbol{\xi}'_2 = 1, \qquad \boldsymbol{\eta}'^{\mathrm{T}}_1\boldsymbol{\xi}'_2 = 0. \tag{40}$$

In view of eqn (38), we see that we do not have to consider all three equations in eqns (40). Since $\boldsymbol{\xi}_1, \boldsymbol{\xi}'_2, \boldsymbol{\eta}'_1, \boldsymbol{\eta}_2$ obtained from eqns (32) and (35) are not unique, we will show how one can obtain a set of vectors so that eqns (40) are satisfied.

Let $\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}'_2, \hat{\boldsymbol{\eta}}'_1, \hat{\boldsymbol{\eta}}_2$ satisfy eqns (32) and (35). They also satisfy eqns (38) and (39). It can be shown with the use of eqns (37) that

$$\left.\begin{aligned}\boldsymbol{\xi}_1 &= k_1\hat{\boldsymbol{\xi}}_1\\ \boldsymbol{\xi}'_2 &= k_2\hat{\boldsymbol{\xi}}'_2+k'_2\hat{\boldsymbol{\xi}}_1\\ \boldsymbol{\eta}'_1 &= k_1\hat{\boldsymbol{\eta}}'_1+k'_2\hat{\boldsymbol{\eta}}_2\\ \boldsymbol{\eta}_2 &= k_2\hat{\boldsymbol{\eta}}_2\end{aligned}\right\} \tag{41}$$

also satisfy eqns (32) and (35) in which k_1, k_2 and k'_2 are arbitrary complex constants which are related by

$$k_2 = k_1+\delta k'_2. \tag{42}$$

Imposition of eqns $(40)_1$, $(40)_2$ and use of eqn (39) lead to

$$k_1^{-2} = \hat{\boldsymbol{\eta}}'^{\mathrm{T}}_1\hat{\boldsymbol{\xi}}_1, \qquad k_2^{-2} = \hat{\boldsymbol{\eta}}_2^{\mathrm{T}}\hat{\boldsymbol{\xi}}'_2. \tag{43}$$

With eqns (43), eqn (38) can be written as

$$k_2^{-2}-k_1^{-2} = \delta\hat{\boldsymbol{\eta}}'^{\mathrm{T}}_1\hat{\boldsymbol{\xi}}'_2. \tag{44}$$

If we solve for (k_2-k_1) from eqn (44) and substitute it into eqn (42) we obtain

$$k'_2 = -k_1^2k_2^2(\hat{\boldsymbol{\eta}}'^{\mathrm{T}}_1\hat{\boldsymbol{\xi}}'_2)/(k_1+k_2). \tag{45}$$

When $\delta \neq 0$, the orthogonal relation of $\hat{\boldsymbol{\xi}}_\alpha, \hat{\boldsymbol{\eta}}_\alpha$ $(\alpha = 1, 2)$ and eqns $(33)_1$ and $(36)_1$ assure us that $\hat{\boldsymbol{\eta}}'^{\mathrm{T}}_1\hat{\boldsymbol{\xi}}_1$ and $\hat{\boldsymbol{\eta}}_2^{\mathrm{T}}\hat{\boldsymbol{\xi}}'_2$ do not vanish. Hence k_1 and k_2 exist. In eqn (45) k_1+k_2 vanishes if $k_1 = -k_2$. However, k_1 obtained from eqn $(43)_1$ is not unique in the sense that if k_1 is a solution so is $-k_1$. The same statement applies to k_2 and one can always choose the signs so that $k_1 = k_2$ instead of $k_1 = -k_2$. Hence k'_2 also exists.

When $\delta = 0$, eqns (43) and (44) can be written as

$$\left.\begin{aligned}k_1^{-2} = k_2^{-2} &= \hat{\boldsymbol{\eta}}'^{\mathrm{T}}_1\hat{\boldsymbol{\xi}}_1 = \hat{\boldsymbol{\eta}}_2^{\mathrm{T}}\hat{\boldsymbol{\xi}}'_2\\ k'_2 &= -k_1^3\hat{\boldsymbol{\eta}}'^{\mathrm{T}}_1\hat{\boldsymbol{\xi}}'_2/2.\end{aligned}\right\} \tag{46}$$

The third equality in (46) comes from eqn (38). The existence of orthonormalized generalized eigenvectors are assured by the theories on non-semisimple matrices[12]. Note that eqns (46) also apply to the case when $\delta \neq 0$ and $k_1^2 = k_2^2$.

With $\xi_1', \xi_2', \eta_1', \eta_2'$ properly orthonormalized, it can be shown that

$$\mathbf{V}'^{T}\mathbf{U}' = \mathbf{I} \tag{47}$$

in which

$$\left.\begin{aligned} \mathbf{U}' &= [\xi_1', \quad \xi_2', \quad \xi_3, \quad \bar{\xi}_1', \quad \bar{\xi}_2', \quad \bar{\xi}_3] \\ \mathbf{V}' &= [\eta_1', \quad \eta_2', \quad \eta_3, \quad \bar{\eta}_1', \quad \bar{\eta}_2', \quad \bar{\eta}_3]. \end{aligned}\right\} \tag{48}$$

In eqns (48) ξ_3 and η_3 are identical to the ones obtained in the last section. If we introduce the 3×3 matrices

$$\mathbf{A}' = [\mathbf{a}_1', \quad \mathbf{a}_2', \quad \mathbf{a}_3], \qquad \mathbf{B}' = [\mathbf{b}_1', \quad \mathbf{b}_2', \quad \mathbf{b}_3] \tag{49}$$

we have

$$\mathbf{U}' = \begin{bmatrix} \mathbf{A}' & \bar{\mathbf{A}}' \\ \mathbf{B}' & \bar{\mathbf{B}}' \end{bmatrix}, \qquad \mathbf{V}' = \mathbf{J}\begin{bmatrix} \mathbf{A}'\mathbf{Y} & \bar{\mathbf{A}}'\bar{\mathbf{Y}} \\ \mathbf{B}'\mathbf{Y} & \bar{\mathbf{B}}'\bar{\mathbf{Y}} \end{bmatrix} \tag{50}$$

where use has been made of eqns (37) and

$$\mathbf{Y} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & \delta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{Y}^{T}. \tag{51}$$

It is useful to know that

$$\mathbf{Y}^{-1} = \begin{bmatrix} -\delta & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (\mathbf{Y}^{-1})^{T} \tag{52}$$

and hence $\mathbf{Y}^{-1} = \mathbf{Y}$ when $\delta = 0$.

As in the last section the product of $\mathbf{U}'$ and $\mathbf{V}'^{T}$ in eqn (47) can be interchanged. That is

$$\mathbf{U}'\mathbf{V}'^{T} = \mathbf{I} \tag{53}$$

or, carrying out the matrix multiplications

$$\left.\begin{aligned} \mathbf{A}'\mathbf{Y}\mathbf{A}'^{T} + \bar{\mathbf{A}}'\bar{\mathbf{Y}}\bar{\mathbf{A}}'^{T} = \mathbf{0} = \mathbf{B}'\mathbf{Y}\mathbf{B}'^{T} + \bar{\mathbf{B}}'\bar{\mathbf{Y}}\bar{\mathbf{B}}'^{T} \\ \mathbf{A}'\mathbf{Y}\mathbf{B}'^{T} + \bar{\mathbf{A}}'\bar{\mathbf{Y}}\bar{\mathbf{B}}'^{T} = \mathbf{I} = \mathbf{B}'\mathbf{Y}\mathbf{A}'^{T} + \bar{\mathbf{B}}'\bar{\mathbf{Y}}\bar{\mathbf{A}}'^{T}. \end{aligned}\right\} \tag{54}$$

This is the modified closure relations for eqns (25). Using the arguments following eqns (25) one is tempted to write

$$\left.\begin{aligned} \mathbf{H} &= 2i\mathbf{A}'\mathbf{Y}\mathbf{A}'^{\mathrm{T}} \\ \mathbf{L} &= -2i\mathbf{B}'\mathbf{Y}\mathbf{B}'^{\mathrm{T}} \\ \mathbf{S} &= i(2\mathbf{A}'\mathbf{Y}\mathbf{B}'^{\mathrm{T}}-\mathbf{I}). \end{aligned}\right\} \tag{55}$$

When $\mathbf{N}$ is non-semisimple, i.e. when $\delta = 0$, the validity of eqns (55) can be established easily by using the relation[8]

$$\langle \mathbf{N} \rangle \boldsymbol{\xi}_\alpha = \pm i \boldsymbol{\xi}_\alpha \tag{56}$$

in which the "+" sign is for $\alpha = 1, 2, 3$, the "$-$" sign is for $\alpha = 4, 5, 6$, and

$$\langle \mathbf{N} \rangle = \begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^{\mathrm{T}} \end{bmatrix}. \tag{57}$$

Equation (56) certainly applies to $\boldsymbol{\xi}'_1$, $\boldsymbol{\xi}_3$ and $\boldsymbol{\xi}'_4$, $\boldsymbol{\xi}_6$. Lothe and Barnett[5] and Chadwick and Smith[6] show that it applies to $\boldsymbol{\xi}'_2$ and $\boldsymbol{\xi}'_5$ also. Therefore, we have

$$\langle \mathbf{N} \rangle \mathbf{U}' = i \begin{bmatrix} \mathbf{A}' & -\bar{\mathbf{A}}' \\ \mathbf{B}' & -\bar{\mathbf{B}}' \end{bmatrix}. \tag{58}$$

If we post-multiply both sides by $\mathbf{V}'^{\mathrm{T}}$ and use eqn (53), we obtain

$$\left.\begin{aligned} \mathbf{H} &= i(\mathbf{A}'\mathbf{Y}\mathbf{A}'^{\mathrm{T}} - \bar{\mathbf{A}}'\bar{\mathbf{Y}}\bar{\mathbf{A}}'^{\mathrm{T}}) \\ \mathbf{L} &= -i(\mathbf{B}'\mathbf{Y}\mathbf{B}'^{\mathrm{T}} - \bar{\mathbf{B}}'\bar{\mathbf{Y}}\bar{\mathbf{B}}'^{\mathrm{T}}) \\ \mathbf{S} &= i(\mathbf{A}'\mathbf{Y}\mathbf{B}'^{\mathrm{T}} - \bar{\mathbf{A}}'\bar{\mathbf{Y}}\bar{\mathbf{B}}'^{\mathrm{T}}). \end{aligned}\right\} \tag{59}$$

Equations (54) and (59) lead to eqns (55).

We will show in the next section that eqns (55) hold also for $\delta \neq 0$. In closing this section we point out that to convert eqns (26) to eqns (55) one cannot simply replace $\mathbf{A}$, $\mathbf{B}$, by $\mathbf{A}'$, $\mathbf{B}'$. The matrix $\mathbf{Y}$ has to be introduced as shown in eqns (55).

4. CONVERSION FROM THE STROH FORMALISM TO THE MODIFIED FORMALISM

If eqns (55) hold for any δ, comparison with eqns (26) suggests that the following conversion relations hold:

$$\left.\begin{aligned} \mathbf{A}\mathbf{A}^{\mathrm{T}} &= \mathbf{A}'\mathbf{Y}\mathbf{A}'^{\mathrm{T}} \\ \mathbf{B}\mathbf{B}^{\mathrm{T}} &= \mathbf{B}'\mathbf{Y}\mathbf{B}'^{\mathrm{T}} \\ \mathbf{A}\mathbf{B}^{\mathrm{T}} &= \mathbf{A}'\mathbf{Y}\mathbf{B}'^{\mathrm{T}}. \end{aligned}\right\} \tag{60}$$

We have proved that eqns (55) and hence eqns (60) hold for $\delta = 0$. It remains to prove that eqns (60) hold for $\delta \neq 0$.

To this end, we will derive the relations between $\boldsymbol{\xi}'_1$, $\boldsymbol{\xi}'_2$, and $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$. Since $\boldsymbol{\xi}'_\alpha$, $\boldsymbol{\eta}'_\alpha$, $\alpha = 1, 2$, are scalar multiples of $\boldsymbol{\xi}_\alpha$, $\boldsymbol{\eta}_\alpha$, we let

$$\left.\begin{aligned} \boldsymbol{\xi}'_1 &= \gamma \boldsymbol{\xi}_1, & \boldsymbol{\xi}'_2 &= \varepsilon \boldsymbol{\xi}_2 \\ \boldsymbol{\eta}'_1 &= \gamma \boldsymbol{\eta}_1, & \boldsymbol{\eta}'_2 &= \varepsilon \boldsymbol{\eta}_2 \end{aligned}\right\} \tag{61}$$

in which eqn (17) has been used and γ, ε are constants to be determined. From eqns $(33)_1$ and $(36)_1$ we have

$$\left.\begin{aligned}\xi_2' &= (\varepsilon\xi_2-\gamma\xi_1)/\delta\\ \eta_1' &= (\varepsilon\eta_2-\gamma\eta_1)/\delta.\end{aligned}\right\} \tag{62}$$

Substituting eqns (61) and (62) into eqns $(40)_1$, $(40)_2$ and making use of eqn (19), we obtain

$$\gamma^2 = -\delta, \qquad \varepsilon^2 = \delta. \tag{63}$$

Recognizing the double solutions for γ and ε in terms of δ, we let

$$\varepsilon = \pm i\gamma, \qquad \gamma^2 = -\delta \tag{64}$$

without identifying which one of the two solutions is for γ. Therefore

$$\left.\begin{aligned}\xi_1' &= \gamma\xi_1\\ \xi_2' &= \gamma^{-1}(\xi_1 \mp i\xi_2)\end{aligned}\right\} \tag{65}$$

and $\mathbf{A}'$ from eqn $(49)_1$ has the expression

$$\mathbf{A}' = [\gamma\mathbf{a}_1, \gamma^{-1}(\mathbf{a}_1 \mp i\mathbf{a}_2), \mathbf{a}_3]. \tag{66}$$

A similar expression can be written for $\mathbf{B}'$. Let

$$\mathbf{E} = \begin{bmatrix}\gamma & \gamma^{-1} & 0\\ 0 & \mp i\gamma^{-1} & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{67}$$

We then have

$$\mathbf{A}' = \mathbf{AE}, \qquad \mathbf{B}' = \mathbf{BE}. \tag{68}$$

By a direct calculation it can be shown that

$$\mathbf{EYE}^{\mathrm{T}} = \mathbf{I}. \tag{69}$$

Equations (68) and (69) lead to the identities in eqns (60). This completes the proof that eqns (60) and hence eqns (55) hold for any δ.

With eqns (60) one can convert relations which are valid for simple or semisimple N to relations for non-semisimple or almost non-semisimple N. For instance, the impedance matrix M is defined as[13]

$$i\mathbf{M} = \mathbf{BA}^{-1}. \tag{70}$$

Since

$$\mathbf{BA}^{-1} = (\mathbf{BB}^{\mathrm{T}})(\mathbf{AB}^{\mathrm{T}})^{-1} \tag{71}$$

using eqns (60) we obtain

$$i\mathbf{M} = \mathbf{B}'\mathbf{A}'^{-1}. \tag{72}$$

This is one of the few relations for which the conversion is achieved by a simple replacement of $\mathbf{A}$, $\mathbf{B}$ by $\mathbf{A}'$, $\mathbf{B}'$.

5. SUM RULES

Several sum rules involving the eigenvalues p_α and eigenvectors ξ_α have been reported in the literature[3, 14, 15]. The sum rules involve the summations of p_α and ξ_α which can be written in matrix notation as one of the following[16]

$$\left.\begin{array}{llll} \mathbf{A}\mathbf{P}^n\mathbf{A}^T, & \mathbf{A}\mathbf{P}^n\mathbf{B}^T, & \mathbf{B}\mathbf{P}^n\mathbf{A}^T, & \mathbf{B}\mathbf{P}^n\mathbf{B}^T \\ \mathbf{A}\mathbf{P}^n\mathbf{A}^{-1}, & \mathbf{A}\mathbf{P}^n\mathbf{B}^{-1}, & \mathbf{B}\mathbf{P}^n\mathbf{A}^{-1}, & \mathbf{B}\mathbf{P}^n\mathbf{B}^{-1} \end{array}\right\} \tag{73}$$

in which n is an integer, positive or negative, and $\mathbf{P}$ is the diagonal matrix

$$\mathbf{P} = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}. \tag{74}$$

Each of the products in (73) can be expressed in terms of the real matrices $\mathbf{H}$, $\mathbf{L}$, $\mathbf{S}$ and $\mathbf{N}_i$, $i = 1, 2, 3$. By a direct calculation, it can be shown that

$$\mathbf{E}\mathbf{P}'\mathbf{E}^{-1} = \mathbf{P}, \qquad \mathbf{E}\mathbf{P}'^n\mathbf{E}^{-1} = \mathbf{P}^n \tag{75}$$

in which $\mathbf{E}$ is given in eqn (67) and, by eqn (69)

$$\mathbf{E}^{-1} = \mathbf{Y}\mathbf{E}^T = \begin{bmatrix} \gamma^{-1} & +i\gamma'^{-1} & 0 \\ 0 & +i\gamma' & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{76}$$

$$\mathbf{P}' = \begin{bmatrix} p_1 & 1 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}. \tag{77}$$

We see that $\mathbf{P}'$ is the Jordan canonical matrix when $p_2 = p_1$. From eqns (68), (69) and (75) we have the following conversion relations:

$$\left.\begin{array}{l} \mathbf{A}\mathbf{P}^n\mathbf{A}^T = \mathbf{A}'\mathbf{P}'^n\mathbf{Y}\mathbf{A}'^T \\ \mathbf{A}\mathbf{P}^n\mathbf{B}^T = \mathbf{A}'\mathbf{P}'^n\mathbf{Y}\mathbf{B}'^T \\ \mathbf{B}\mathbf{P}^n\mathbf{A}^T = \mathbf{B}'\mathbf{P}'^n\mathbf{Y}\mathbf{A}'^T \\ \mathbf{B}\mathbf{P}^n\mathbf{B}^T = \mathbf{B}'\mathbf{P}'^n\mathbf{Y}\mathbf{B}'^T \end{array}\right\} \tag{78}$$

and

$$\left.\begin{array}{l} \mathbf{A}\mathbf{P}^n\mathbf{A}^{-1} = \mathbf{A}'\mathbf{P}'^n\mathbf{A}'^{-1} \\ \mathbf{A}\mathbf{P}^n\mathbf{B}^{-1} = \mathbf{A}'\mathbf{P}'^n\mathbf{B}'^{-1} \\ \mathbf{B}\mathbf{P}^n\mathbf{A}^{-1} = \mathbf{B}'\mathbf{P}'^n\mathbf{A}'^{-1} \\ \mathbf{B}\mathbf{P}^n\mathbf{B}^{-1} = \mathbf{B}'\mathbf{P}'^n\mathbf{B}'^{-1}. \end{array}\right\} \tag{79}$$

Equations (78) suggest that $\mathbf{P}'^n\mathbf{Y}$ is symmetric. Indeed, it is readily shown that

$$\mathbf{P}'^{n} = \begin{bmatrix} p_1^n & y_n & 0 \\ 0 & p_2^n & 0 \\ 0 & 0 & p_3^n \end{bmatrix} \tag{80}$$

where

$$y_n = \sum_{k=1}^{n} p_2^{n-k} p_1^{k-1} = \begin{cases} (p_2^n - p_1^n)/(p_2 - p_1), & \text{if } p_2 \neq p_1 \\ n p_1^{n-1}, & \text{if } p_2 = p_1. \end{cases} \tag{81}$$

Hence

$$\mathbf{P}'^{n}\mathbf{Y} = \begin{bmatrix} y_n & p_2^n & 0 \\ p_2^n & \delta p_2^n & 0 \\ 0 & 0 & p_3^n \end{bmatrix} = (\mathbf{P}'^{n}\mathbf{Y})^{\mathrm{T}}. \tag{82}$$

6. SEPARATION OF ξ_2' INTO $\mathbf{a}_2'$ AND $\mathbf{b}_2'$

The Stroh eigen-relation was in fact based on the earlier version, eqns (5) and (8) proposed by Eshelby *et al.*[7]. Thus instead of finding the 6-vector $\boldsymbol{\xi}$ from eqn (11) one could find the 3-vectors $\mathbf{a}$ and $\mathbf{b}$ from eqns (5) and (8). This may have some advantages in a numerical calculation because not only the matrix $\mathbf{D}$ is smaller than $\mathbf{N}$, one does not have to find the inverse $\mathbf{T}^{-1}$ as shown in eqns (13). When $\mathbf{N}$ is non-semisimple, so is $\mathbf{D}(p)$ of eqn (5). This means that when $p_2 = p_1$, $\mathbf{a}_2 = \mathbf{a}_1$. To modify eqns (5) and (8) for the cases when $\mathbf{D}(p)$ is non-semisimple or almost non-semisimple, we follow the derivation of eqns (32). We obtain

$$\left.\begin{aligned} &\mathbf{D}(p_1)\mathbf{a}_1' = \mathbf{0} \\ &\mathbf{D}(p_2)\mathbf{a}_2' + \{(\mathbf{R}+\mathbf{R}^{\mathrm{T}}) + (p_1+p_2)\mathbf{T}\}\mathbf{a}_1' = \mathbf{0}. \end{aligned}\right\} \tag{83}$$

As to the modification of eqn (8), we have

$$\left.\begin{aligned} &\mathbf{b}_1' = (\mathbf{R}^{\mathrm{T}} + p_1\mathbf{T})\mathbf{a}_1' = -\left(\frac{1}{p_1}\mathbf{Q} + \mathbf{R}\right)\mathbf{a}_1' \\ &\mathbf{b}_2' = (\mathbf{R}^{\mathrm{T}} + p_2\mathbf{T})\mathbf{a}_2' + \mathbf{T}\mathbf{a}_1' = -\left(\frac{1}{p_2}\mathbf{Q} + \mathbf{R}\right)\mathbf{a}_2' + \frac{1}{p_1 p_2}\mathbf{Q}\mathbf{a}_1'. \end{aligned}\right\} \tag{84}$$

Equations (83) and (84) provide $\mathbf{a}_1'$, $\mathbf{a}_2'$, $\mathbf{b}_1'$ and $\mathbf{b}_2'$ which form the components of ξ_1' and ξ_2'. One then finds η_1', η_2' from eqns (37) and orthonormalize the eigenvectors as outlined in Section 3. To complete the system, one finds $\mathbf{a}_3$, $\mathbf{b}_3$ of ξ_3 from eqns (5), (8), η_3 from eqn (17) and normalize ξ_3 using eqn (19).

For isotropic materials we have $p_1 = p_2 = i$, $\mathbf{a}_1 = \mathbf{a}_2$, and the outlined procedure leads to

$$\mathbf{A}' = \begin{bmatrix} k_1 & -i\kappa k_1 & 0 \\ ik_1 & -\kappa k_1 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \tag{85}$$

$$\mathbf{B}' = \mu \begin{bmatrix} 2ik_1 & k_1 & 0 \\ -2k_1 & -ik_1 & 0 \\ 0 & 0 & ik_3 \end{bmatrix} \tag{86}$$

$$k_1^2 = \frac{1}{8\mu(1-\nu)}, \qquad k_3^2 = \frac{-i}{2\mu} = \frac{(1-i)^2}{4\mu}, \qquad \kappa = \frac{3-4\nu}{2} \tag{87}$$

where μ and ν are, respectively, the shear modulus and Poisson's ratio. With $\mathbf{A}'$, $\mathbf{B}'$ given by eqns (85), (86) and $\mathbf{Y}$ by eqn (51) with $\delta = 0$, eqns (55) provide $\mathbf{H}$, $\mathbf{L}$ and $\mathbf{S}$ for isotropic materials. The non-zero elements of $\mathbf{H}$, $\mathbf{L}$ and $\mathbf{S}$ are

$$\left.\begin{aligned} H_{11} &= H_{22} = \frac{3-4\nu}{4\mu(1-\nu)}, \quad H_{33} = \frac{1}{\mu} \\ L_{11} &= L_{22} = \frac{\mu}{1-\nu}, \quad L_{33} = \mu \\ S_{21} &= -S_{12} = \frac{1-2\nu}{2(1-\nu)}. \end{aligned}\right\} \tag{88}$$

This agrees with the results obtained by using the integral formalism in Ref. [8].

7. CONCLUDING REMARKS

The modified sextic formalism presented here applies to any matrix $\mathbf{N}$ which is simple, almost non-semisimple or non-semisimple. The formalism is particularly useful when $\mathbf{N}$ is non-semisimple or almost non-semisimple. Thus instead of the integral formalism[8], eqns (55) offer an alternate way of obtaining the three real matrices $\mathbf{H}$, $\mathbf{L}$, $\mathbf{S}$ when $\mathbf{N}$ is non-semisimple or almost non-semisimple.

We did not consider the possibility of a non-semisimple $\mathbf{N}$ in which $p_1 = p_2 = p_3$ and $\xi_1 = \xi_2 = \xi_3$. We have not seen such an example and it appears unlikely that there exists a real material which leads to $p_1 = p_2 = p_3$ and $\xi_1 = \xi_2 = \xi_3$. For isotropic materials $p_1 = p_2 = p_3 = i$ but $\xi_1 = \xi_2 \neq \xi_3$ and hence the modified formalism applies to isotropic materials as shown in the last section.

REFERENCES

1. A. N. Stroh, Dislocations and cracks in anisotropic elasticity. *Phil. Mag.* **7**, 625–646 (1958).
2. A. N. Stroh, Steady state problems in anisotropic elasticity. *J. Math. Phys.* **41**, 77–103 (1962).
3. K. Nishioka and J. Lothe, Isotropic limiting behavior of the six-dimensional formalism of anisotropic dislocation theory and anisotropic Green's function theory, (I) sum rules and their applications. *Phys. Stat. Sol. B* **51**, 645–656 (1972).
4. K. Nishioka and J. Lothe, Isotropic limiting behavior of the six-dimensional formalism of anisotropic dislocation theory and anisotropic Green's function theory, (II) perturbation theory on the isotropic N-matrix. *Phys. Stat. Sol. B* **52**, 45–54 (1972).
5. J. Lothe and D. M. Barnett, On the existence of surface-wave solution for anisotropic elastic half-space with free surface. *J. Appl. Phys.* **47**, 428–433 (1976).
6. P. Chadwick and G. D. Smith, Foundations of the theory of surface waves in anisotropic elastic materials. *Adv. Appl. Mech.* **17**, 303–376 (1977).
7. J. D. Eshelby, W. T. Read and W. Shockley, Anisotropic elasticity with applications to dislocation theory. *Acta Metall.* **1**, 251–259 (1953).
8. D. M. Barnett and J. Lothe, Synthesis of the sextic and the integral formalism for dislocation, Green's functions and surface waves in anisotropic elastic solids. *Phys. Norv.* **7**, 13–19 (1973).
9. D. M. Barnett and J. Lothe, An image force theorem for dislocations in anisotropic bicrystals. *J. Phys. F* **4**, 1618–1635 (1974).
10. D. M. Barnett and J. Lothe, Line force loadings on anisotropic half-space and wedges. *Phys. Norv.* **8**, 13–22 (1975).
11. T. C. T. Ting, Explicit solution and invariance of the singularities at an interface crack in anisotropic composites. *Int. J. Solids Structures* **22**, 965–983 (1986).
12. M. C. Pease, III, *Methods of Matrix Algebra*. Academic Press, New York (1965).
13. K. A. Ingebrigsten and A. Tonning, Elastic surface waves in crystals. *Phys. Rev.* **184**(3), 942–951 (1969).
14. D. J. Bacon, D. M. Barnett and R. O. Scattergood, The anisotropic continuum theory of lattice defects. *Prog. Mater. Sci.* **23**, 51–262 (1978).
15. R. J. Asaro, J. P. Hirth, D. M. Barnett and J. Lothe, A further synthesis of sextic and integral theories for dislocations and line forces in anisotropic media. *Phys. Stat. Sol. B* **60**, 261–271 (1973).
16. T. C. T. Ting, Some identities and the structure of $\mathbf{N}_i$ in the Stroh formalism of anisotropic elasticity. *Q. Appl. Math.* 1988, in press.